Cofinite Induced Subgraph Nim

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Nim

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- Game played with n heaps of beans
- Players alternate removing any positive number of beans from any one heap
- When all heaps are empty the next player has no moves and loses

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- Players alternate removing any positive number of beans from any one heap
- When all heaps are empty the next player has no moves and loses
- $\{x_1, x_2, \cdots x_n\}$ is the position where pile i has x_i beans
- Can move from $\{x_1, x_2, \cdots x_n\}$ to $\{y_1, y_2, \cdots y_n\}$ if $y_i < x_i$ for a single value of i and $y_i = x_i$ for all others

Game Graphs

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Game Graphs:

- Possibly infinite directed graph
- Vertices are positions
- An edge is drawn from u to v if there is a move from u to v
- If there is an edge from u to v, then v is a child of u
- No loops
- Starting from any given vertex, only finitely many other vertices are reachable by any sequence of moves

P- and N-positions

- P-position: Previous player has a winning strategy
- N-position: Next player has a winning strategy
- Every position of an impartial combinatorial game can be classified as either a P-position or N-position

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- P-position: Previous player has a winning strategy
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- Every position of an impartial combinatorial game can be classified as either a P-position or N-position
- Perfect play only requires identifying P-positions

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- Every child of a P-position is an N-position
- Every N-position has a P-position child
- These properties uniquely define set of P- and N- positions

Why Nim?

Sprague-Grundy Value:

- Let G and H be positions of possibly different games.
- $G + H$ is game in which players can choose to move in either G or H on each turn
- $\{x_1, \cdots x_n\} + \{y_1 \cdots y_m\} = \{x_1, \cdots x_n, y_1 \cdots y_m\}$

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- There exists a unique x such that $G + \{x\}$ is a P-position
- x is called the Sprague-Grundy value of G, or $sg(G)$

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- x is called the Sprague-Grundy value of G, or $sg(G)$
- $sg(G) = 0$ iff G is a P-position
- $sg(G_1 + G_2 + \cdots + G_n) = sg(\{sg(G_1), sg(G_2), \cdots, sg(G_n)\})$
- This allows us to analyze sums of games by converting the individual summands to Nim heaps!

Solution to Nim

To compute $a \oplus b$:

- Convert a and b to binary
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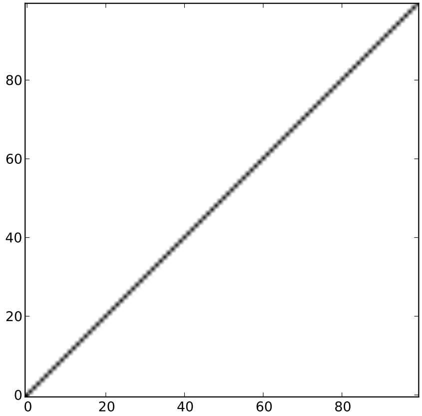
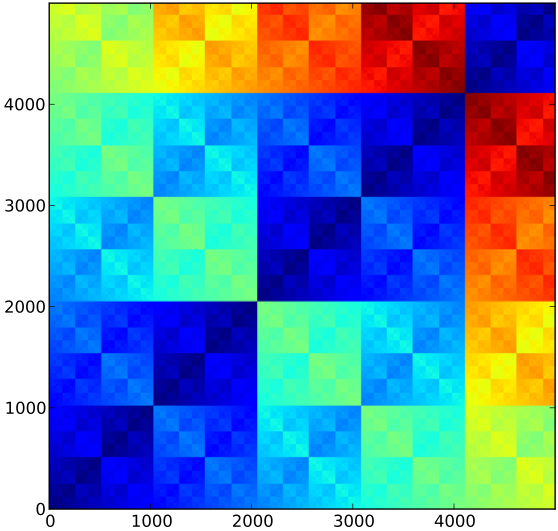
- Add a and b without carrying

$\{x_1, x_2, \cdots x_n\}$ is a P-position iff $x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0$

Visualizing 2-heap Nim



Visualizing 3-heap Nim



Cofinite Induced Subgraph Games

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Cofinite Induced Subgraph Games

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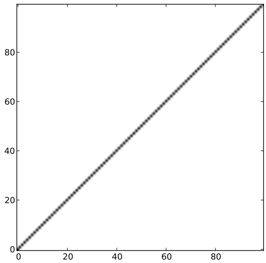
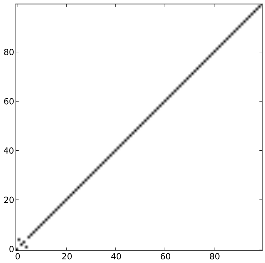
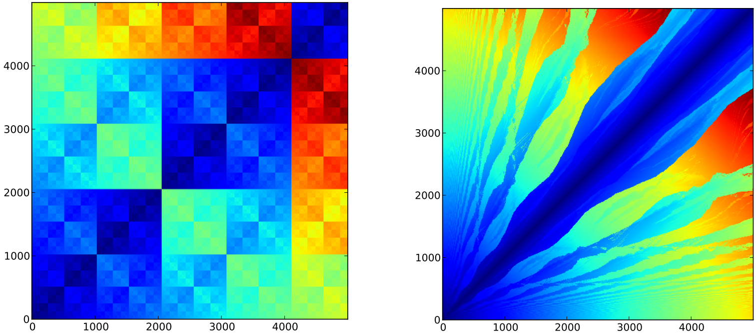
- We will be interested in restricting movement to any position in some finite set F
- This corresponds to taking a cofinite induced subgraph of the game graph
- Such games are called "Cofinite Induces Subgraph Games," or "CIS Games"
- Allows us to determine what properties of games are independent of the endgame

2-Heap CIS-Nim





3-Heap CIS-Nim



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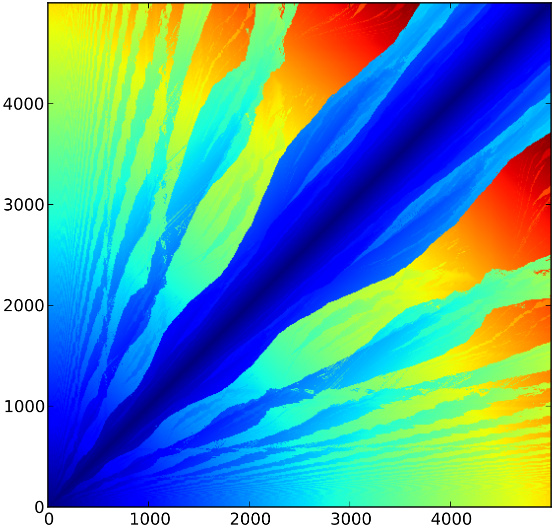

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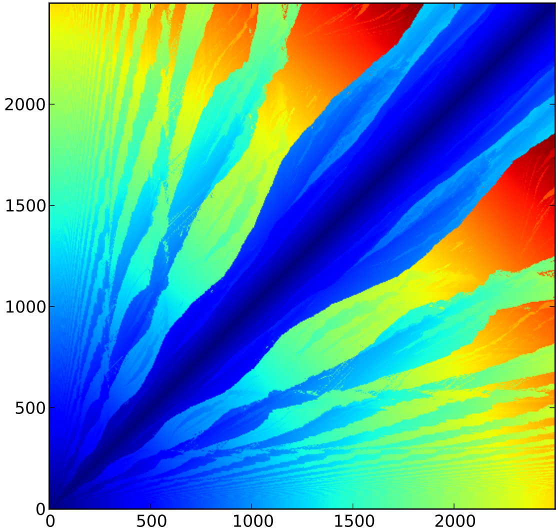

Theorem

For any nonnegative integers x and y there is a unique z such that $\{x, y, z\}$ is a P-position. This value of z satisfies the inequality $z \leq x + y + |F|$.

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- Consider all positions of the form $\{x, y, z\}$ with $z \leq x + y + |F|$
- There are $x + y + |F| + 1$ of them
- At most $|F|$ of them are in F
- At most x are N-positions with P-position child, $\{x', y, z\}$
- At most y are N-positions with P-position child, $\{x, y', z\}$

A Tighter Bound

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Let c be equal to the largest element of any position in F. If $\{x, y, z\}$ is a P-position with $z > 2c + |F|$, then $z \leq x + y$.

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- Either $x > c$ or $y > c$

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- Either $x > c$ or $y > c$
- Consider the set of all points of the form $\{x, y, z\}$ with $z \leq x + y$
- There are $x + y + 1$ of them, none of which are in F

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- There are $x + y + 1$ of them, none of which are in F
- At most x are N-positions with P-position child, $\{x', y, z\}$
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- Corollary: For all $n > 2c + |F|$, $\{n, n, 0\}$ is a P-position

Periodicity

Theorem

For any x, there exists a p and a q such that for any $y > q$, $\{x, y, z\}$ is a P-position if and only if $\{x, y + p, z + p\}$ is a P-position.

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- Proven by induction on x
- Base case is previous Corollary

Period Two Scale Invariance

Theorem

Let $\pi(n)$ denote the number of P-positions of the form $\{x, y, z\}$, with x, y, and z all less than n. For any positive integer n, $\lim_{k\to\infty} \frac{\pi(n2^k)}{(n2^k)^2}$ converges to a nonzero constant.

S

- Let S be the set of all ordered pairs (x, y) such that there is a P-position of the form $\{x, y, z\}$ with $x > y > z$
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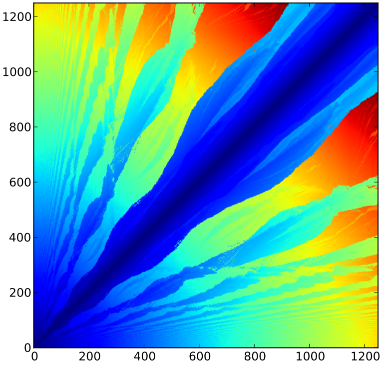

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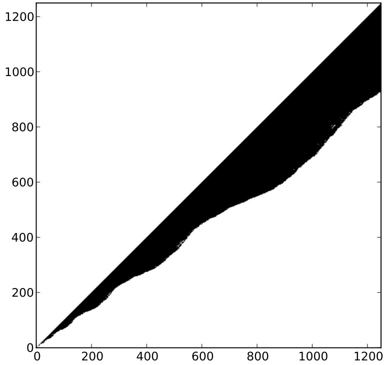
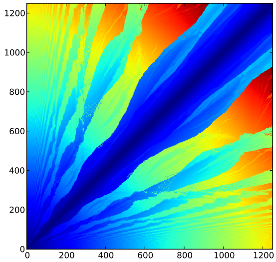
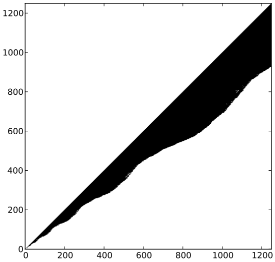


A Flawed Proof





- Let $y = f(x)$ be the curve on the lower boundary of S

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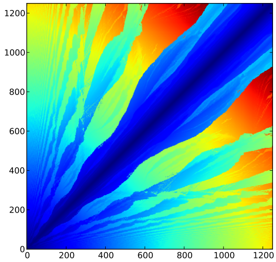

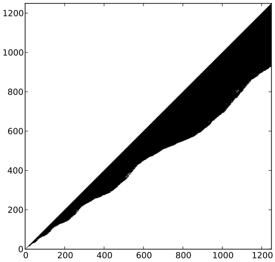

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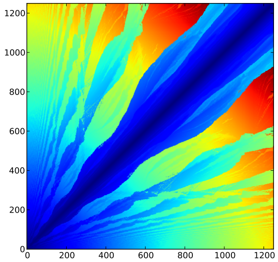

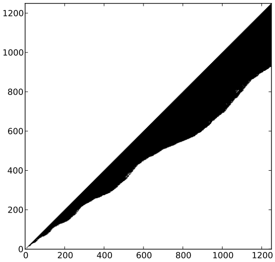

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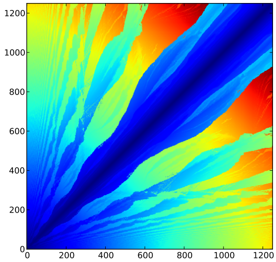

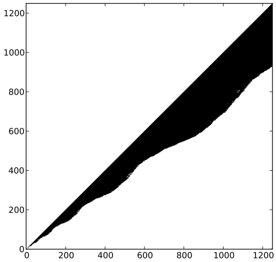

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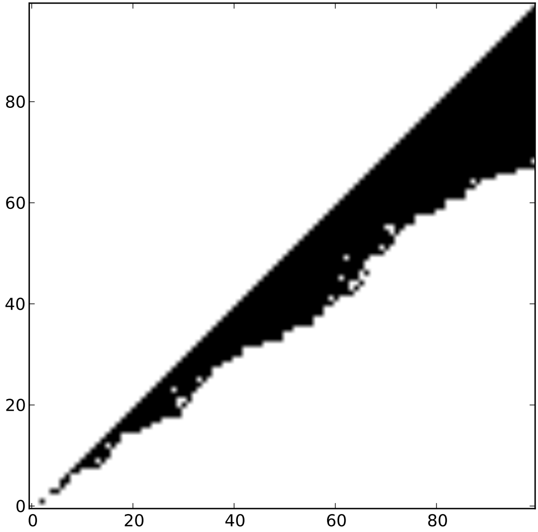
- $\int_0^{2n} x - f(x)dx = 2\int_0^n 2x - f(2x)dx = 4\int_0^n x - f(x)dx$

The "Hole" in the Proof



$b(x, y)$ and $r(x, y)$

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Lemma

For all sufficiently large n, $r(n, n) + 2b(n, n) + 1 = n$.

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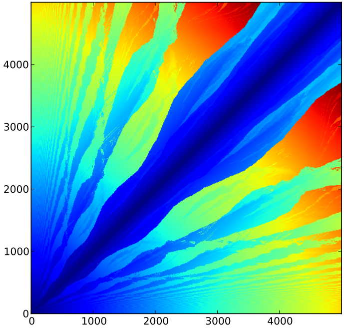

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For all $x > y > c$*, if* $(x, y) \notin S$ *then* $b(x, y) \geq r(x, y)$.

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Proof Sketch:

- $r(x, y)$ is the number of $(x', y) \in S$ with $x' > x$
- There are therefore $r(x, y)$ P-positions of the form $\{x', y, z\}$ with $x' > x > y > z$.

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- These P-position children must be of the form $\{x, y', z\}$ with $y' < y$
- Each one will contribute one point of the form (x, y') with $y' \leq y$ to S, contributing 1 to $b(x, y)$

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Lemma

For any sufficiently large m, there exists a bijection ϕ from S_m to S_{m+1} such that if $\phi(x_1, y_1) = (x_2, y_2)$, then $x_1 - y_1 \geq x_2 - y_2$ and $x_1 - 2y_1 \geq x_2 - 2y_2$.

Completing the Proof Strategy

Theorem

Let $\pi(n)$ denote the number of P-positions of the form $\{x, y, z\}$, with x, y, and z all less than n. For any positive integer n, $\lim_{k\to\infty} \frac{\pi(n2^k)}{(n2^k)^2}$ converges to a nonzero constant.

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- Assume BWOC that it does not converge
- There therefore exists an open ball (p, q), such that $\frac{\pi(n2^k)}{(n2^k)^2} > q$ and $\frac{\pi(n2^k)}{(n2^k)^2} < p$, each for infinitely many values of k

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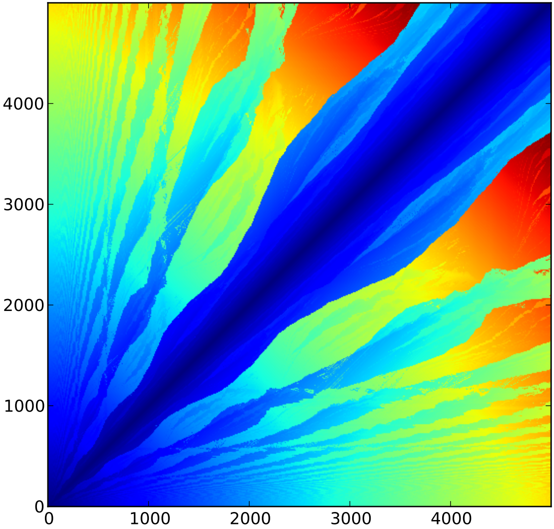
- Because points in S_m can only move in the same direction, we can set up a potential which will eventually be depleted, stopping us from making these movements.

What about the Background?



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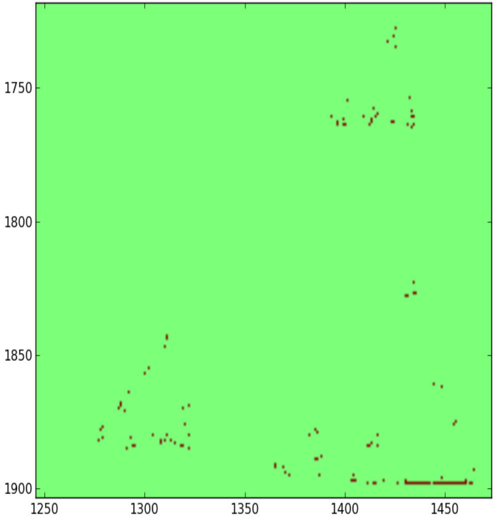

Questions?